

# Sphalerons and large order behaviour of perturbation theory in lower dimension

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## Abstract

Sphalerons – unstable static solutions of classical field equations in  $(d+1)$ -dimensional space-time – may be viewed as euclidean solutions in  $d$  dimensions. We discuss their role in the large order asymptotics of the perturbation theory. Specifically, we calculate their contribution to the large order behaviour of the ground state energy in a quantum mechanical model. When the number of negative modes is odd, single sphaleron contribution dominates, while this contribution vanishes when the number of negative modes is even. These results are confirmed by numerical calculations.

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# 1 Introduction

Many smooth, finite action classical solutions to euclidean field equations describe semiclassical tunneling transitions. Ordinary instantons in Yang-Mills theory [1] and their analogs (instantons in  $(1+1)$ -dimensional abelian Higgs model, non-linear sigma model, etc. ) correspond to transitions between degenerate classical vacua [2, 3, 4], while bounce solutions [5] are relevant to false vacuum decay. One of the differences between these two types of solutions is that the number of negative modes is zero in the former case and one in the latter.

In this paper we discuss yet another type of euclidean finite action solutions, namely, sphalerons [6, 7]. Sphalerons are usually considered as static solutions in  $(d+1)$ -dimensional field theories where they determine the height of a barrier between different vacua. However, they may be viewed also as (time-dependent) solutions to euclidean field equations in  $d$ -dimensional space-time. In this context we will call these solutions "sphinstantons". We are interested in their role in  $d$ -dimensional field theories.

Examples of sphinstantons are:

- i) original sphaleron [6, 7] viewed as the euclidean solution in 3-dimensional Yang-Mills-Higgs theory;
- ii) sphaleron in 2-dimensional non-linear sigma model with explicit symmetry breaking [8] interpreted as the euclidean solution in quantum mechanics of a particle moving on a two-dimensional sphere in homogeneous gravitational field;
- iii) the solution  $I^*$  in 4-dimensional euclidean Yang-Mills-Higgs theory, whose existence has been recently advocated in refs. [9, 10]; etc.

Like bounces, sphinstantons have negative modes (sphalerons in  $(d+1)$  dimensions are unstable). At first sight, this implies, that these negative modes signalize the vacuum instability, and the sphinstantons describe the semiclassical vacuum decay.

However, in known examples there are no classical field configurations with energy smaller than the energy of the trivial vacuum. Thus, the trivial vacuum is stable (at least semiclassically), so the vacuum decay interpretation of sphinstantons is not possible. The formal reason for this is the absence of a non-trivial turning point (the slice  $t=\text{const}$  at which time derivatives of the fields is zero at all  $\mathbf{x}$ ), as opposed to the bounce solutions.

Besides describing tunneling, instantons play another role in field theory and quantum mechanics. Namely, bounces and instanton – anti-instanton pairs contribute to the high orders of perturbation theory [11, 12, 13, 14] (for a review see [15]). The contribution of the instanton – anti-instanton pair to the  $k$ -th order of perturbation theory is proportional to

$$k!/(2S_I)^k, \quad (1)$$

where  $S_I$  is the action of the instanton. The fact that instanton – anti-instanton pairs, rather than single instantons, contribute to eq. (1) is related to the absence of negative modes around the instantons. One may expect that sphinstantons also contribute to high orders of perturbation theory, and their contribution behaves as

$$k!/S_{SI}^k, \quad (2)$$

if the number of negative modes is odd, and

$$k!/(2S_{SI})^k, \quad (3)$$

if the number of negative modes is even (in the latter case the contribution should again come from sphinstanton pairs). Here  $S_{SI}$  is the sphinstanton action (equal to the static energy of the sphaleron in  $(d+1)$  dimensions).

The purpose of this paper is to check these expectations, eqs. (2) and (3), in a simple quantum mechanical model of a particle moving on a sphere in external potential. As pointed out above, sphinstanton in this model is the sphaleron in

2-dimensional non-linear sigma model with explicit symmetry breaking term. We describe the model and its sphinstanton in sect. 2. In sect. 3 we present detailed calculations of the high orders of perturbation theory for the ground state energy, making use of the technique analogous to ref. [15]. In fact, we consider in sect. 3 a particle on  $n$ -dimensional sphere, where  $n$  need not be integer. We will see that at all  $n$  except for  $n = 1, 3, 5, \dots$ , the asymptotics is given by eq. (2), while at integer odd  $n$  (when the sphinstanton has even number of negative modes), the one-sphinstanton contribution vanishes, so that the asymptotics is given by eq. (3). In sect. 4 we present the results of numerical calculations of the high orders of perturbation theory which agree with the analytical results of sect. 3. Sect. 5 contains concluding remarks.

## 2 The model and sphinstanton

We consider quantum mechanics of a particle with unit mass moving on  $n$ -dimensional sphere in external potential. The classical euclidean action is

$$S = \int dt [(d\mathbf{N}/dt)^2 + V(N_1)],$$

where  $\mathbf{N} = (N_1, \dots, N_{n+1})$ ,  $\mathbf{N}^2 = 1$ . We consider the potentials depending only on one coordinate  $N_1$ , and assume that it has one minimum at  $N_1 = -1$ , where  $V(-1) = 0$ ,  $V'(-1) = 1$ . Then the classical ground state is the point

$$\mathbf{N}_0 = (-1, 0, 0, \dots, 0)$$

and its classical energy is zero. Notice that both in classical and quantum mechanics the ground state is unique, and, therefore, stable.

Sphinstanton in this model (cf. ref. [8]) up to  $O(n)$  rotations of coordinates  $N_2, \dots, N_{n+1}$  is

$$N_1 = \cos \theta(t), N_2 = \sin \theta(t),$$

where  $\theta(t)$  is determined by the following relation

$$t = \int_0^\theta \frac{d\theta}{\sqrt{2V(\cos \theta)}}.$$

As  $t$  runs from  $-\infty$  to  $+\infty$ , the particle makes a loop around the sphere from the south pole through the north pole back to the south pole (see fig. 1). The action for the sphinstanton is

$$S_{SI} = \int_{-\pi}^{\pi} \sqrt{2V(\cos \theta)} d\theta$$

It is obvious that there exist  $(n-1)$  negative modes around the sphinstanton (cf. ref. [8]), which correspond to the shifts of the loop in  $(n-1)$  directions, see fig. 1.

The evaluation of the contributions of classical solutions to high orders of perturbation theory is by now standard (see ref. [15]). In this paper we consider perturbation theory for the ground state energy, the perturbation parameter being  $\hbar$ ,

$$E_0 = \hbar E^{(0)} + \hbar^2 E^{(1)} + \dots + \hbar^{k+1} E^{(k)} + \dots \quad (4)$$

Roughly speaking,  $E^{(k)}$  is given by the integral

$$E^{(k)} \sim \int \frac{d\hbar D \mathbf{N}}{2\pi i \hbar^{k+1}} \exp(-S/\hbar),$$

where the integrals are to be taken by the saddle-point method. The contribution of the sphinstanton to  $E^{(k)}$  is then

$$E^{(k)} \sim \int \frac{d\hbar}{2\pi i \hbar^{k+1}} D^{-1/2} \exp(-S_{SI}/\hbar),$$

where  $D$  is the determinant of fluctuations around the sphinstanton. To make real contribution to  $E^{(k)}$ , the sphinstanton should have odd number of negative modes, so that  $D^{1/2}$  is pure imaginary. Then the asymptotic behaviour of  $E^{(k)}$  is given by eq. (2). When the number of negative modes is even, the contribution of a single sphinstanton should vanish (otherwise it would be pure imaginary, in contradiction

to the general property of reality of  $E^{(k)}$ ). So, we expect that the behaviour of  $E^{(k)}$  at large  $k$  is

$$E^{(k)} \propto k!/S_{SI}^k, \quad \text{even } n,$$

$$E^{(k)} \propto k!/(2S_{SI})^k, \quad \text{odd } n.$$

We will check this property by numerical calculations in sect. 4, and now we turn to actual calculation of the sphinstanton contribution to  $E^{(k)}$  at arbitrary (not necessarily integer) dimensionality of the sphere.

### 3 Sphinstanton contribution to high orders of perturbation theory

In this section we evaluate the sphinstanton contribution to the perturbative expansion of the ground state energy. Since the potential depends only on  $\theta$ , the ground state has zero angular momentum with respect to other  $(n-1)$  angular variables, i.e., the wave function of the ground state depends only on  $\theta$ . Thus, we consider the s-wave Hamiltonian

$$H = -\frac{\hbar^2}{2} \frac{d^2}{d\theta^2} - \frac{\hbar^2}{2} (n-1) \frac{\cos \theta}{\sin \theta} \frac{d}{d\theta} + V(\cos \theta) \quad (5)$$

The wave function of the ground state should be non-singular at the south and north poles, so we require

$$\frac{d\psi}{d\theta}(\theta = 0) = 0 \quad (6)$$

$$\frac{d\psi}{d\theta}(\theta = \pi) = 0 \quad (7)$$

The Hamiltonian (5) and boundary conditions (6), (7) make sense at arbitrary, not necessarily integer  $n$ . So, we may generalize the problem by defining the ground state of a particle on fractionally dimensional sphere as the lowest energy state of the Hamiltonian (5).

We are interested in  $k$ -th order of the perturbation theory for the ground state energy,  $E^{(k)}$ , see eq. (4). To evaluate this term, it is convenient to consider the statistical sum

$$Tr \exp(-\beta H/\hbar) = \int_{\theta(0)=\theta(\beta)} D\theta \exp(-S/\hbar)$$

The  $k$ -th order term in the expansion of the statistical sum is formally written as

$$(Tr \exp(-\beta H/\hbar))^{(k)} = \int \frac{d\hbar D\theta}{2\pi i \hbar^{k+1}} \exp(-S/\hbar), \quad (8)$$

where the contour of integration over  $\hbar$  runs around the origin counterclockwise.

Consider now the sphinstanton contribution to the integral

$$(Tr \exp(-\beta H/\hbar))_{SI} = \int_{\theta(0)=\theta(\beta)} D\theta \exp(-S/\hbar) \quad (9)$$

We will see that this contribution behaves as follows,

$$(Tr \exp(-\beta H/\hbar))_{SI} = \beta \exp(-\beta n/2) i B \hbar^{-\alpha} \exp(-S_{SI}/\hbar) \quad (10)$$

where  $\alpha$  and  $B$  are some functions of  $n$ . From eq. (8) it follows that the sphinstanton gives rise to the following asymptotic behaviour at large  $k$

$$(Tr \exp(-\beta H/\hbar))^{(k)} = \beta \exp(-\beta n/2) \frac{k! k^{\alpha-1} B}{2\pi S_{SI}^{k+\alpha}} \quad (11)$$

On the other hand, at large  $\beta$  the statistical sum is saturated by the ground state, so that

$$Tr \exp(-\beta H/\hbar) = \exp(-\beta(E^{(0)} + \hbar E^{(1)} + \dots + \hbar^k E^{(k)} + \dots)) \quad (12)$$

where  $E^{(0)} = n/2$ . One then expands eq. (12) in  $\hbar$  and makes use of the factorial growth, eq. (11), to obtain (cf. ref. [15])

$$E^{(k)} = -\frac{k! k^{\alpha-1} B}{2\pi S_{SI}^{k+\alpha}} \quad (13)$$

So, our main problem is to evaluate the sphinstanton contribution to the functional integral (9).

Before the actual calculation of the sphinstanton contribution, we make a few comments concerning the integral (9). First, we wish to make use of the saddle point technique. The Hamiltonian (5) has, in fact, a term of order  $\hbar$ , so it is convenient to write the euclidean action in the following form, up to terms of order  $\hbar^2$ ,

$$S = S_0 + \hbar S_1,$$

where

$$S_0 = \int dt(-ip_\theta \dot{\theta} + H_0(p_\theta, \theta)),$$

$$S_1 = \int dt H_1(p_\theta, \theta),$$

where

$$H_0 = p_\theta^2/2 + V(\cos \theta),$$

$$H_1 = -i \frac{n-1}{2} \frac{\cos \theta}{\sin \theta} p_\theta,$$

where  $p_\theta$  is the momentum conjugate to  $\theta$ . We will consider saddle points of the action  $S_0$  and treat  $\exp(-S_1)$  as the pre-exponential factor. Second, the points  $\theta = 0$  and  $\theta = \pi$  are singular points where the semiclassical approximation does not work. When treating the regions close to these points, we will have, in particular, to take into account the boundary conditions (6), (7). Finally, we first study finite values of  $\beta$ . The relevant classical euclidean trajectory is then periodic in time with the period  $\beta$ . We take the variable  $\theta$  to belong to the interval  $\theta \in (0, \pi)$ , so the function  $\theta(t)$  has discontinuous time derivatives at  $\theta = 0$  and  $\theta = \pi$  (see fig. 2). This will not make a serious problem because we will anyway treat the points  $\theta = 0$  and  $\theta = \pi$  "exactly".

Let us consider the vicinities of the points  $\theta = \pi$  and  $\theta = 0$  in more detail. We choose some points  $\theta_r = \pi - r$ ,  $\theta_s = s$ , that separate semiclassical and non-semiclassical regions, and take  $r$  and  $s$  small enough. Every trajectory close to the classical solution has the form shown in fig. 2. We may choose the origin of time in such a way that the trajectory passes the point  $\theta_r$ , running towards  $\theta = \pi$ , at  $t = 0$ .



Then it passes the point  $\theta_r$  again at some time  $t = \tau_1$ , and then passes the point  $\theta_s$  twice at  $t = \tau_1 + \tau_2$  and  $t = \tau_1 + \tau_2 + \tau_3$ , as shown in fig. 2. Notice that at large  $\beta$ , the point spends almost all euclidean "time" near  $\theta = \pi$ , so that  $\tau_1$  is close to  $\beta$ , while  $\tau_2$ ,  $\tau_3$  and  $(\beta - \tau_1 - \tau_2 - \tau_3)$  remain finite in the limit  $\beta \rightarrow \infty$ . Guided by this observation, we insert the identity

$$\begin{aligned} \beta \int d\tau_1 d\tau_2 d\tau_3 \delta(\theta(0) - \theta_r) \delta(\theta(\tau_1) - \theta_r) \delta(\theta(\tau_1 + \tau_2) - \theta_s) \delta(\theta(\tau_1 + \tau_2 + \tau_3) - \theta_s) \\ \times \dot{\theta}(0) \dot{\theta}(\tau_1) \dot{\theta}(\tau_1 + \tau_2) \dot{\theta}(\tau_1 + \tau_2 + \tau_3) = 1 \end{aligned}$$

into the integral (9). This identity is valid when integrated with the weight invariant under time translations, and the factor  $\beta$  accounts for the time translational invariance. The resulting integral is then expressed through Green's function

$$G(\theta_1, \theta_2; t_2 - t_1) = \int_{\theta(t_1)=\theta_1, \theta(t_2)=\theta_2} \exp(-S/\hbar) D\theta \quad (14)$$

Namely, we obtain

$$\begin{aligned} (Tr \exp(-\beta H/\hbar))_{SI} = \beta \int d\tau_1 d\tau_2 d\tau_3 \dot{\theta}(0) \dot{\theta}(\tau_1) \dot{\theta}(\tau_1 + \tau_2) \dot{\theta}(\tau_1 + \tau_2 + \tau_3) \\ \times G(\theta_r, \theta_r; \tau_1) G(\theta_r, \theta_s; \tau_2) G(\theta_s, \theta_s; \tau_3) G(\theta_s, \theta_r; \beta - \tau_1 - \tau_2 - \tau_3) \end{aligned} \quad (15)$$

The integral (14) for Green's functions  $G(\theta_r, \theta_s; \tau_2)$  and  $G(\theta_s, \theta_r; \beta - \tau_1 - \tau_2 - \tau_3)$  should be calculated by the saddle-point (semiclassical) technique. The semiclassical expression for Green's function  $G(\theta_r, \theta_s; \tau_2)$  has the following form [16, 17]

$$\begin{aligned} G(\theta_r, \theta_s; \tau_2) = \frac{\exp(-\int_{\tau_1}^{\tau_1+\tau_2} H_1 d\tau)}{\sqrt{2\pi\hbar}} \left( \frac{\partial\theta(\tau_1 + \tau_2, \theta_r, \dot{\theta}(\tau_1))}{\partial\dot{\theta}(\tau_1)} \right)^{-1/2} \\ \times \exp(-S(\theta_r, \theta_s; \tau_2)/\hbar) \end{aligned} \quad (16)$$

where  $\theta(\tau, \theta_r, \dot{\theta}(\tau_1))$  is the classical trajectory which starts at  $\theta_r$  at  $\tau = \tau_1$  with the velocity  $\dot{\theta}(\tau_1)$ ; after taking the derivative in eq. (16) one should choose  $\dot{\theta}(\tau_1)$  in such a way that the trajectory passes  $\theta = \theta_s$  at  $t = \tau_1 + \tau_2$ ;  $S(\theta_r, \theta_s; \tau_2)$  is the classical

action along the latter trajectory. A completely analogous expression may be written for  $G(\theta_s, \theta_r; \beta - \tau_1 - \tau_2 - \tau_3)$ .

On the other hand, Green's functions  $G(\theta_r, \theta_r; \tau_1)$  and  $G(\theta_s, \theta_s; \tau_3)$  cannot be calculated by the saddle point technique: the regions  $\theta > \theta_r$  and  $\theta < \theta_s$  are not semi-classical. Consider first  $G(\theta_r, \theta_r; \tau_1)$ . Since  $\tau_1 \approx \beta$  at large  $\beta$ , we may approximate this Green's function by the ground state contribution to its expansion through the eigenfunctions of the Hamiltonian,

$$G(\theta_r, \theta_r; \tau_1) = r^{n-1} \psi_0(r) \psi_0^*(r) \exp(-\tau_1 E_0)$$

Furthermore, since  $r$  is small, the Hamiltonian at  $\pi > \theta > \theta_r = \pi - r$  coincides with its quadratic part,

$$H_0 = -\frac{\hbar^2}{2} \frac{d^2}{dr^2} - \frac{\hbar^2}{2} \frac{n-1}{r} \frac{d}{dr} + \frac{r^2}{2}$$

so that the ground state wave function is

$$\psi_0(r) = \left( \frac{2}{\hbar^{n/2} \Gamma(n/2)} \right)^{1/2} \exp(-r^2/2\hbar)$$

Note that this wave function obeys the boundary condition (7) automatically. We obtain finally

$$G(\theta_r, \theta_r; \tau_1) = \frac{2r^{n-1} \exp(-r^2/\hbar)}{\hbar^{n/2} \Gamma(n/2)} \exp(-\tau_1 n/2).$$

It is now straightforward to integrate over  $\tau_1$  and  $\tau_2$  in eq. (15). This integral is again of the saddle-point nature, with the relevant exponent being

$$\exp \left( -\frac{S(\theta_r, \theta_s; \tau_2) + S(\theta_s, \theta_r; \beta - \tau_1 - \tau_2 - \tau_3)}{\hbar} \right)$$

In the limit  $\beta \rightarrow \infty$ , the saddle point in  $\tau_1$  and  $\tau_2$  is obtained when the two trajectories connecting  $\theta_r$  and  $\theta_s$  have zero energy, i.e., when the classical trajectory is the sphinstanton. The pre-exponential factors are obtained by making use of the following relations:

i) the duration of the trajectory connecting  $\theta_r$  and  $\theta_s$  at energy  $E$  is

$$\tau_2(E) = - \int_{\theta_r}^{\theta_s} \frac{d\theta}{\sqrt{2(E + V(\cos \theta))}}$$

ii) the expression entering eq. (16) is

$$\frac{\partial \theta(\tau_1 + \tau_2, \theta_r; \dot{\theta}(\tau_1))}{\partial \dot{\theta}(\tau_1)} = -\dot{\theta}(\tau_1 + \tau_2) \dot{\theta}(\tau_1) \left( \frac{d\tau_2(E)}{dE} \right)_{E=0}$$

iii) the second derivative of the action with respect to  $\tau_2$  is

$$\frac{\partial^2 S(\theta_r, \theta_s; \tau_2)}{\partial \tau_2^2} = - \left( \frac{1}{d\tau_2(E)/dE} \right)_{E=0}$$

iv) the integrals  $\int H_1 d\tau$  along the two parts of the sphinstanton trajectory cancel each other, because  $H_1$  is linear in momentum  $p_\theta$ ;

v) by the explicit calculation of  $\theta(\tau)$  for the sphinstanton one finds at small  $\theta_s$  and small  $r = \pi - \theta_r$ ,

$$r \exp(-\tau_1/2) = \pi \exp(-\beta/2) \exp \int_0^\pi d\theta \left( \frac{1}{\sqrt{2V(\cos \theta)}} - \frac{1}{\pi - \theta} \right)$$

Collecting all pre-exponential factors we write the following expression for the sphinstanton contribution (15) after the integration over  $\tau_1$  and  $\tau_2$ ,

$$\begin{aligned} (Tr \exp(-\beta H/\hbar))_{SI} &= \frac{2\beta \exp(-\beta n/2)}{\hbar^{n/2} \Gamma(n/2)} \left( \pi \exp \left[ \int_0^\pi d\theta \left( \frac{1}{\sqrt{2V(\cos \theta)}} - \frac{1}{\pi - \theta} \right) \right] \right)^n \\ &\times \int d\tau_3 \sqrt{2V(\cos \theta_s)} G(\theta_s, \theta_s; \tau_3) \end{aligned} \quad (17)$$

Notice that the above technique can be used for obtaining high order behaviour of the perturbation theory in simpler models. We show in Appendix that the known results [11, 15] for  $O(n)$ -symmetric single well potentials are indeed reproduced in this way.

The remaining part of the calculation is the evaluation of the "sphinstanton contribution" to Green's function  $G(\theta_s, \theta_s; \tau_3)$ . At  $0 < \theta < \theta_s \ll 1$ , the Hamiltonian is

approximated as follows

$$H = -\frac{\hbar^2}{2} \frac{d^2}{d\theta^2} - \frac{\hbar^2}{2} \frac{n-1}{\theta} \frac{d}{d\theta} + E_{NP},$$

where  $E_{NP} = V(\cos \theta = 1)$  is the potential energy at the north pole. The exact Green's function of this operator, which obeys the boundary condition (6), is

$$G(\theta_2, \theta_1; \tau) = \frac{2\pi^{(n-1)/2} \theta_2^{n-1}}{\Gamma((n-1)/2) (2\pi\hbar\tau)^{n/2}} \exp\left(-\frac{E_{NP}\tau}{\hbar} - \frac{\theta_1^2 + \theta_2^2}{2\hbar\tau}\right) \\ \times \int_{-1}^1 d\lambda \exp\left(\frac{\theta_1\theta_2\lambda}{\hbar\tau}\right) (1-\lambda^2)^{(n-3)/2}$$

The extrema of the exponent in the integral over  $\lambda$  are end points,  $\lambda = \pm 1$ . We keep the contribution of  $\lambda = -1$  only. This contribution corresponds to trajectories that are reflected by the point  $\theta = 0$ : indeed, the  $\theta$ -dependent term in the exponent at  $\lambda = -1$  is

$$(\theta_1 + \theta_2)^2 / (2\hbar\tau)$$

which is precisely the action for such trajectories. To evaluate the contribution of  $\lambda = -1$ , we change the integration contour as shown in fig. 3 and obtain the "sphinstanton contribution"

$$(G(\theta_s, \theta_s; \tau_3))_{SI} = -\frac{2i \sin(\pi(n-3)/2)}{\sqrt{2\pi\hbar\tau_3}} \exp\left(-\frac{E_{NP}\tau_3}{\hbar} - \frac{2\theta_s^2}{\hbar\tau_3}\right)$$

We need the integral of this expression over  $\tau_3$  (see eq. (17)). The integration is straightforwardly performed by the saddle point technique; we find

$$\int d\tau_3 \sqrt{2E_{NP}} (G(\theta_s, \theta_s; \tau_3))_{SI} = -2i \cos(\pi n/2)$$

Inserting this expression into eq. (17), we obtain the sphinstanton contribution to the integral (8) in the form (10) where

$$\alpha = n/2,$$

$$B = -\frac{4\pi^n}{\Gamma(n/2)} \cos(\pi n/2) \exp \left( n \int_0^\pi d\theta \left( \frac{1}{\sqrt{2V(\cos \theta)}} - \frac{1}{\pi - \theta} \right) \right)$$

According to eq. (13), the high order behaviour of the ground state energy is finally

$$E^{(k)} = \frac{k!k^{n/2-1}}{S_{SI}^{k+n/2}} \cos(\pi n/2) \frac{2\pi^{n-1}}{\Gamma(n/2)} \exp \left( n \int_0^\pi d\theta \left( \frac{1}{\sqrt{2V(\cos \theta)}} - \frac{1}{\pi - \theta} \right) \right) \quad (18)$$

An interesting feature of this formula is that it is zero at integer odd  $n$ , i.e., the sphinstanton does not contribute to the high order behaviour of the perturbation theory at these  $n$ . This confirms our expectations (sect. 2) based on counting of the negative modes.

## 4 Numerical results

In this section we confirm the results of sect. 3 by numerical calculations of the perturbative expansion of the ground state energy. The numerical study is based on the following method [18]. The wave function of the ground state is expanded in  $\hbar$ ,

$$\psi = \sum \psi^{(k)} \hbar^k$$

where

$$\begin{aligned} \psi^{(k)} &= \sum_{l=0}^{2k} A_{k,l} x^{2l} \exp(-x^2/2), \\ x &= (\pi - \theta)/\sqrt{\hbar} \end{aligned}$$

The coefficients  $A_{k,l}$  obey recursive relations which enable one to develop the numerical procedure.

We perform numerical calculations for the particular form of the potential,

$$V(\cos \theta) = \cos \theta + 1 \quad (19)$$

This choice corresponds to a particle moving on a sphere in homogeneous gravitational field. The recursive relations in this case have the following form

$$\begin{aligned}
& -(l+1)(2l+n)A_{k,l+1} + \sum_{m=2}^l (-1)^{m+1} A_{k-m+1,l-m}/m! - \sum_{m=1}^k E^{(m)} A_{k-m,l} \\
& + 2lA_{k,l} + \sum_{m=1}^l f_m A_{k-m,l-m} - 2 \sum_{m=1}^l f_m (l-m+1) A_{k-m,l-m+1} = 0
\end{aligned} \tag{20}$$

where

$$f_m = -\frac{(n-1)2^{2m}|B_{2m}|}{2(2m)!}$$

are the coefficients of the expansion of

$$\frac{(n-1)x \cos x}{2 \sin x} = \sum f_k x^{2k}$$

( $B_{2m}$  are Bernoulli numbers), and by definition

$$A_{0,0} = 1,$$

$$A_{k,0} = 0, \quad k > 0$$

$$A_{k,l} = 0, \quad l < 0, \quad l > 2k$$

As before,  $E^{(k)}$  is the  $k$ -th coefficient in the expansion of the ground state energy.

Eq. (20) defines the numerical procedure for evaluation of both  $A_{k,l}$  and  $E^{(k)}$ . We are interested in the latter quantity, which should be compared to eq. (18). For the particular choice of the potential, eq. (19), we have

$$S_{SI} = 8$$

and

$$\int_0^\pi d\theta \left( \frac{1}{\sqrt{2V(\cos \theta)}} - \frac{1}{\pi - \theta} \right) = \ln(4/\pi)$$

So, we find from eq. (19) that the following quantity,

$$D_k = \frac{8^k E^{(k)}}{k! k^{n/2-1}} \tag{21}$$

tends to constant as  $k \rightarrow \infty$ ,

$$\lim_{k \rightarrow \infty} D_k = D = \frac{2^{n/2+1} \cos(\pi n/2)}{\pi \Gamma(n/2)} \quad (22)$$

The results of our calculations are shown in figs. 4-6. In fig. 4a,b,c we plot  $D_k$  at various  $n$ . It is clear that they indeed tend to the values given by eq. (22) which are shown by dashed lines.

In fig. 5 we plot  $D_k$  versus  $k$  (in logarithmic scale) at integer odd  $n$ . We see from fig. 5 that sphinstantons indeed do not contribute to the high orders of perturbation theory at these  $n$  ( $D_k$  exponentially tend to zero at large  $k$ ).

Fig. 6 shows that the behaviour of  $E^{(k)}$  at large  $k$ , and  $n$  close to odd number is consistent with the sum of one sphinstanton and sphinstanton – anti-sphinstanton contributions,

$$E^{(k)} = C_1 k^{\alpha_1} \cos(\pi n/2) k! / (S_{SI}^k) + C_2 k^{\alpha_2} k! / (2S_{SI})^k$$

where  $C_{1,2}(n)$  and  $\alpha_{1,2}(n)$  are regular at  $n = 1, 3, 5, \dots$ . Namely, at relatively small  $k$  the ratio

$$F_k = E^{(k+1)} / (k E^{(k)}) \quad (23)$$

is close to  $1/(2S_{SI}) = 1/16$  (the sphinstanton pair dominates) but at

$$k \gg |\ln(\cos(\pi n/2))|$$

the single sphinstanton wins, and  $F_k$  becomes equal to  $1/S_{SI} = 1/8$ .

Thus, our numerical results show that it is indeed the single sphinstanton that determines the high order behaviour of the perturbation theory at  $n \neq 1, 3, 5, \dots$ , while at integer odd  $n$  the behaviour is quite different and consistent with the dominance of the sphinstanton – anti-sphinstanton pair.

## 5 Conclusions

It often happens that classical solutions determining the factorial growth of the perturbation series have physical significance (describe tunneling) either in the original theory, or in the theory at unphysical value of the coupling constant [11, 12, 13, 14, 15]. Sphinstantons break this "rule" : while contributing to large orders of perturbation theory, they do not have any other apparent physical meaning. Another specific feature of these solutions is that either single sphinstantons or sphinstanton – anti-sphinstanton pairs contribute to the asymptotics of perturbative expansions depending on whether the number of negative modes around them is odd or even. We have demonstrated this property by rather indirect analytical calculations, as well as by the numerical study. The explicit mechanism which makes the determinant about a single sphinstanton to vanish in the case of odd number of negative modes remains to be understood.

We are indebted to T. Banks, Yu. A. Kubyshin, D. T. Son and P. G. Tinyakov for helpful discussions.

## Appendix

In this Appendix we illustrate the technique of sect. 3 by a simpler example of  $O(n)$ -symmetric system with the potential  $V(r)$  shown in fig. 7,  $V(r) \simeq r^2/2$  if  $r \rightarrow 0$ . The radial Hamiltonian reads

$$H = -\frac{\hbar^2}{2} \frac{d^2}{dr^2} - \frac{n-1}{2r} \hbar^2 \frac{d}{dr} + V(r)$$

The relevant classical solution is bounce that starts at euclidean time  $t = -\infty$  at  $r = 0$ , reaches the turning point  $r_+$  at  $t = 0$  and returns to  $r = 0$ . The action for the bounce is

$$S_B = 2 \int_0^{r_+} dr \sqrt{2V(r)}$$



Repeating the arguments of sect. 3 we write the contribution of the bounce in the form analogous to eq. (17),

$$(Tr \exp(-\beta H/\hbar))_B = \left[ \int d\tau \sqrt{2V(R)} G(R, R; \tau) \right] \times \frac{2\beta \exp(-\beta n/2)}{\hbar^{n/2} \Gamma(n/2)} \exp(-S_B/\hbar) \left( r_+ \exp \left( \int_0^{r_+} dr [1/\sqrt{2V(r)} - 1/r] \right) \right)^n \quad (24)$$

where  $R$  is close to  $r_+$ . To evaluate Green's function  $G(R, R; \tau)$  we point out that near  $r_+$ , the potential is well approximated by the linear function. Retaining only leading terms in  $\hbar$ , we write the Hamiltonian at  $R < r < r_+$  in the following way,

$$H = -\frac{\hbar^2}{2} \frac{d^2}{dy^2} + ay$$

where  $y = r_+ - r$ . The exact Green's function at coinciding arguments for this Hamiltonian is

$$G(y_R, y_R; \tau) = \frac{1}{\sqrt{2\pi\hbar\tau}} \exp(-S(y_R, y_R; \tau)/\hbar)$$

where

$$S(y_R, y_R; \tau) = a\tau y_R - \frac{a^2\tau^3}{24} - \frac{4\sqrt{2a}y_R^{3/2}}{3}$$

$$y_R = r_+ - r$$

(We do not write the factor due to the "quantum" part of the Hamiltonian,  $H_1 = -\frac{n-1}{2r} \hbar^2 \frac{d}{dr}$ , because this factor cancels out precisely in the same way as in sect. 3.)

Performing the integration over  $\tau$  by the saddle point technique, we obtain at small  $y_R$

$$\int d\tau \sqrt{2V(R)} G(R, R; \tau) = i \quad (25)$$

According to eqs. (10), (13), the large order asymptotics of the perturbation series for the ground state energy is found from eqs. (24), (25)

$$E^{(k)} = -\frac{k! k^{n/2-1}}{\pi \Gamma(n/2) S_B^{k+n/2}} \left( r_+ \exp \int_0^{r_+} dr [1/\sqrt{2V(r)} - 1/r] \right)^n \quad (26)$$

At  $n = 1$  this result coincides with the expression given in ref. [15]. We know only one previous calculation at arbitrary  $n$ , namely, for quartic potential [11]  $V(r) = r^2/2 - r^4$ . In that case we have  $r_+ = 1/\sqrt{2}$ ,  $S_B = 1/3$ , and

$$\int_0^{1/\sqrt{2}} dr [1/\sqrt{2V(r)} - 1/r] = \ln 2$$

So we find from eq. (26)

$$E^{(k)} = -\frac{k!k^{n/2-1}}{\pi\Gamma(n/2)}3^{k+n/2}2^{n/2}$$

which coincides with the result of ref. [11]. Thus, our technique reproduces the known asymptotics.

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## Figure captions

**Fig. 1 :** Sphinstanton ( solid line ) at  $n = 2$  and its deformation along the negative mode ( dashed line ) .

**Fig. 2 :** Classical trajectory at finite but large  $\beta$ .

**Fig. 3 :** Integration contour in complex  $\lambda$  - plane.

**Fig. 4 :** Coefficients  $D_k$  defined by eq. (21) , as functions of the order of perturbation theory,  $k$  , at various dimensionality of the sphere,  $n$ :

(a):  $n = 1.1$  to  $1.9$ ;

(b):  $n = 3.1$  to  $3.9$ ;

(c):  $n = 2, 4, 6, 8, 10$ .

Dashed lines are asymptotic values, eq. (22).

**Fig. 5 :**  $D_k$  versus  $k$  at integer odd dimensionality of the sphere.

**Fig. 6 :** The ratio (23) as function of the order of perturbation theory  $k$  at the dimensionalities of the sphere close to  $n = 1$ . The curves correspond to

1)  $n - 1 = 10^{-3}$  ;

2)  $n - 1 = 10^{-5}$  ;

3)  $n - 1 = 10^{-7}$  ;

4)  $n - 1 = 10^{-9}$  ;

5)  $n - 1 = 10^{-11}$  ;

6)  $n - 1 = 10^{-13}$  .

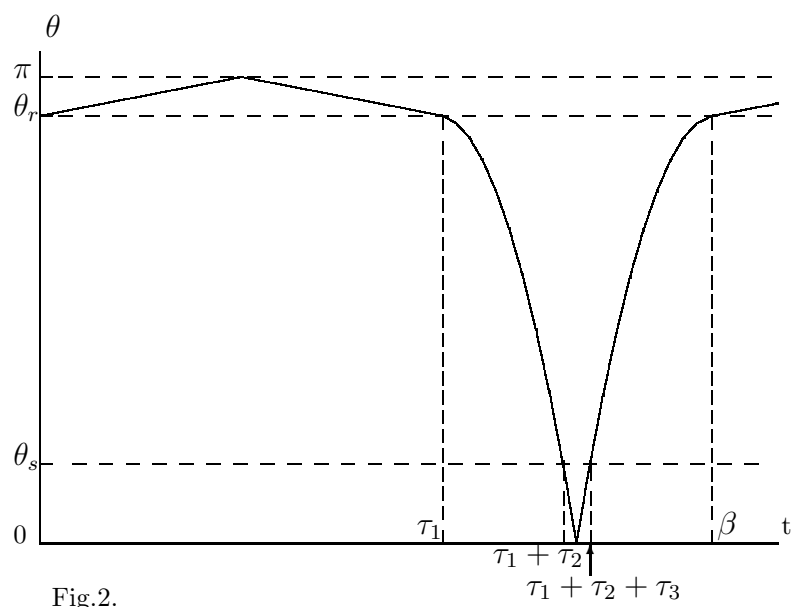


Fig.2.

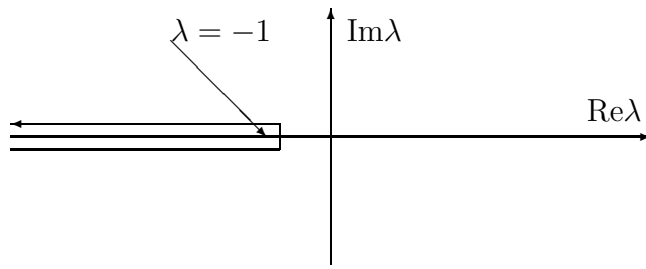


Fig.3

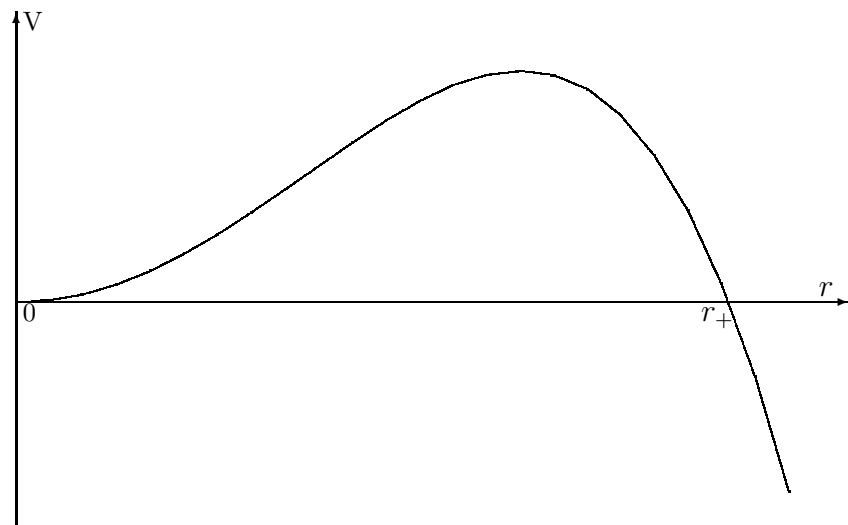


Fig.7

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